The Zel'dovich-type approximation for an inhomogeneous universe in general relativity: second-order solutions

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Abstract

The gravitational instability of inhomogeneities in the expanding universe is studied by the relativistic second-order approximation. Using the tetrad formalism we consider irrotational dust universes and get equations very similar to those given in the Lagrangian perturbation theory in Newtonian cosmology. Neglecting the cosmological constant and assuming a flat background model we give the solutions of the nonlinear dynamics of cosmological perturbations. We present the complete second-order solutions, which extend and improve earlier works.

I. INTRODUCTION

Gravitational instability and structure formation in the universe is an important topic of cosmological research. By using N-Body codes it is possible to follow the general nonlinear evolution of initially small perturbations numerically, but an understanding of what has happened between input and output can often better be gained by analytical treatments. Various analytical approaches have been compared with the numerical results statistically [1,2] and it has turned out that the celebrated Zel'dovich approximation [3] gives the best fit to the numerical treatment. Buchert [4] presented the Lagrangian perturbative approximation to first-order based on Newtonian theory. This work was extended to second-order [5] and even to third-order [6], giving some new useful information about self-gravitating systems. But these and most other analytical treatments are Newtonian approaches, which are valid only for perturbations on scales much smaller than the horizon size. On super-horizon scales instead one needs a relativisic approach. The pioneer was Lifshitz [7] with his linearized theory on the basis of general relativity, which was extended to the second-order by Tomita [8–10]. A Lagrangian relativistic approximation to the second-order based on fluid flow equations was given by Matarrese, Pantano and Saez [11,12]. Parry, Salopek and Stewart [13] presented the nonlinear solution of the Hamilton-Jacobi equation for general relativity, using the spatial gradient expansion technique [14] and reproduced the Zel'dovich approximation. The "Higher-order Zel'dovich approximation" is discussed in Croudace et al. [15] and Salopek, Stewart and Croudace [16].

In this paper, we give an alternative approach, which extends a tetrad-based Zel'dovich-type approximation by Kasai [17]. We derive the fully general relativistic equations very similar to those given in the Newtonian case, which are solved in a flat background model without the cosmological constant by an iteration method. The complete solutions are compared with previous work and it is found that they include all these results.

This paper is organized as follows. In Sec. II we present the basic relativistic equations and introduce the tetrad formalism. In Sec. III the perturbative approach is presented and

the solutions up to second-order are given. In Sec. IV we compare our results with previous works. Sec. V contains conclusions. In the appendices we explain our gauge condition and present the complete second-order solutions, including the decaying and the coupling mode. Units are chosen so that $c = 1$. Indices $\mu, \nu, \cdots$ and $a, b, \cdots$ run from 0 to 3 and indices $i, j, \cdots$ run from 1 to 3.

II. EXPOSITION OF THE METHOD

In this section, we summarize a general relativistic treatment to describe the non-linear evolution of an inhomogeneous universe [17–19]. The models we consider contain irrotational dust with density ρ and four-velocity u^μ (and possibly a cosmological constant Λ). Neglecting the fluid pressure and the vorticity is a reasonable assumption in a cosmological context. In comoving synchronous coordinates, the line element can be written in the form

$$ds^2 = -dt^2 + g_{ij} dx^i dx^j \tag{2.1}$$

with $u^\mu = (1, 0, 0, 0)$. Then the Einstein equations read

$$\frac{1}{2}\left[{}^3R^i{}_i + \left(K^i{}_i\right)^2 - K^i{}_j K^j{}_i \right] = 8\pi G\rho + \Lambda \ , \tag{2.2}$$

$$K^i{}_{j\|i} - K^i{}_{i\|j} = 0 \ , \tag{2.3}$$

$$\dot{K}^i{}_j + K^k{}_k K^i{}_j + {}^3R^i{}_j = (4\pi G\rho + \Lambda)\, \delta^i{}_j \ , \tag{2.4}$$

where ${}^3R^i{}_j$ is the three-dimensional Ricci-tensor,

$$K^i{}_j = \frac{1}{2} g^{ik} \dot{g}_{jk} \tag{2.5}$$

is the extrinsic curvature, $\|$ denotes the covariant derivative with respect to the three metric g_{ij}, and an overdot $(\dot{})$ denotes $\partial/\partial t$. The energy equation $u_\mu T^{\mu\nu}{}_{;\nu} = 0$ gives

$$\dot{\rho} + \rho K^i{}_i = 0 \ , \tag{2.6}$$

with the solution

$$\rho = \rho(t_{in}, \mathbf{x}) \frac{\sqrt{\det\left[g_{ij}(t_{in}, \mathbf{x})\right]}}{\sqrt{\det\left[g_{ij}(t, \mathbf{x})\right]}} \ . \tag{2.7}$$

The evolution equation for the Ricci curvature is obtained in the form [17]

$${}^3\dot{R}^i{}_j + 2K^i{}_k\,{}^3R^k{}_j = K^i{}_{k\|j}{}^{\|k} + K^k{}_j{}^{\|i}{}_{\|k} - K^i{}_j{}^{\|k}{}_{\|k} - K^k{}_k{}^{\|i}{}_{\|j} \ . \tag{2.8}$$

Let us introduce the scale factor function $a(t)$ which satisfies the following equation

$$2a\ddot{a} + \dot{a}^2 + k - \Lambda a^2 = 0 \ , \tag{2.9}$$

where the curvature constant k takes the value of $+1, 0, -1$ for closed, flat, and open spaces, respectively. (Eq. (2.9) is obtained from the Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\rho_b + \frac{\Lambda}{3} \tag{2.10}$$

and its derivative with respect to time.) If the spacetime is exactly Friedmann-Lemaître-Robertson-Walker (FLRW), then we have

$$K^i{}_j = \frac{\dot{a}}{a}\delta^i{}_j, \qquad {}^3R^i{}_j = 2\frac{k}{a^2}\delta^i{}_j \qquad \text{for FLRW} \ . \tag{2.11}$$

Therefore, the deviations from the FLRW models due to inhomogeneity are expressed by the peculiar part of the extrinsic curvature

$$V^i_j \equiv K^i{}_j - \frac{\dot{a}}{a}\delta^i{}_j \ , \tag{2.12}$$

which represents the deviation from the uniform Hubble expansion, and the deviation of the curvature tensor

$$\mathcal{R}^i{}_j \equiv a^2\,{}^3R^i{}_j - 2k\delta^i{}_j \ . \tag{2.13}$$

Using these quantities, Eqs. (2.3), (2.4), and (2.8) are rewritten as

$$V^i_{j|i} - V^i_{i|j} = 0 \ , \tag{2.14}$$

$$\dot{V}^i_j + \left(3\frac{\dot{a}}{a} + V^k_k\right)V^i_j + \frac{1}{a^2}\left(\mathcal{R}^i{}_j - \frac{1}{4}\mathcal{R}^k{}_k\,\delta^i{}_j\right) = \frac{1}{4}\left\{(V^k_k)^2 - V^k_\ell V^\ell_k\right\}\delta^i{}_j \ , \tag{2.15}$$

$$\dot{\mathcal{R}}^i{}_j + 2V^i_\ell \mathcal{R}^\ell{}_j + 4k\, V^i_j = V^i{}_{\ell|j}{}^{|\ell} + V^\ell_j{}^{|i}{}_{|\ell} - V^i_j{}^{|\ell}{}_{|\ell} - V^\ell_\ell{}^{|i}{}_{|j} \,, \tag{2.16}$$

where $|$ denotes the covariant derivative with respect to the conformally transformed three metric $\gamma_{ij} \equiv a^{-2} g_{ij}$.

The procedure essential to develop the relativistic Zel'dovich-type approximation [17] is to introduce the following orthonormal tetrad

$$g_{\mu\nu} = \eta_{(a)(b)} \bar{e}^{(a)}_\mu \bar{e}^{(b)}_\nu \tag{2.17}$$

with

$$\bar{e}^{(0)}_\mu = u_\mu = (-1, 0, 0, 0)\,, \qquad \bar{e}^{(\ell)}_\mu = \left(0, \bar{e}^{(\ell)}_i\right) \equiv \left(0, a(t)\, e^{(\ell)}_i\right) \quad \text{for } \ell = 1, 2, 3. \tag{2.18}$$

The spatial basis vectors are parallelly transported along each fluid line, i.e.,

$$\bar{e}^{(\ell)}_{\mu;\nu}\, u^\nu = 0\,. \tag{2.19}$$

In our choice of the tetrad components, it reads

$$\dot{e}^{(\ell)}_i = V^j_i\, e^{(\ell)}_j \qquad \text{or} \qquad V^i_j = e^i_{(\ell)}\, \dot{e}^{(\ell)}_j\,. \tag{2.20}$$

Using Eqs. (2.15), (2.16) and (2.20), we obtain the following key equation

$$\frac{\partial}{\partial t}\left[a^3\left(\ddot{e}^{(\ell)}_i + 2\frac{\dot{a}}{a}\dot{e}^{(\ell)}_i - 4\pi G \rho_b e^{(\ell)}_i\right)\right] = a\left(P^{(\ell)}_i + Q^{(\ell)}_i + S^{(\ell)}_i\right)\,, \tag{2.21}$$

where

$$P^{(\ell)}_i = \frac{\partial}{\partial t}\left\{\frac{a^2}{4}\left[\left(V^k_k\right)^2 - V^k_n V^n_k\right] e^{(\ell)}_i - a^2\left(V^k_k V^j_i - V^j_k V^k_i\right) e^{(\ell)}_j\right\}\,, \tag{2.22}$$

$$Q^{(\ell)}_i = \left(V^j_k \mathcal{R}^k{}_i + \frac{1}{4} V^j_i \mathcal{R}^k{}_k - \frac{1}{2}\delta^j{}_i\, V^k_n \mathcal{R}^n{}_k\right) e^{(\ell)}_j\,, \tag{2.23}$$

and

$$S^{(\ell)}_i = \left[V^j_i{}^{|k}{}_{|k} + V^k_k{}^{|j}{}_{|i} - V^j_{k|i}{}^{|k} - V^k_i{}^{|j}{}_{|k} + k\left(3V^j_i - V^k_k \delta^j{}_i\right)\right] e^{(\ell)}_j\,. \tag{2.24}$$

Note that the left-hand-side of Eq. (2.21) is already linearized with respect to $e^{(\ell)}_i$, and all terms on the right-hand-side, except $S^{(\ell)}_i$, are manifestly nonlinear quantities. It has, therefore, a form suitable for solving it perturbatively by iteration. It should also be stressed that we have not used any approximation methods in deriving Eq. (2.21). Our treatment here is fully nonlinear and exact.

III. PERTURBATIVE APPROACH

In this section, we solve perturbatively the key equation Eq. (2.21) by an iteration method.

A. The background

The background ($V^i_{\ j} = 0, \mathcal{R}^i_{\ j} = 0$) solution is characterized by

$$\dot{e}^{(\ell)}_{\ i} = 0 \ , \quad \text{i.e.,} \quad e^{(\ell)}_{\ i} = e^{(\ell)}_{\ i}(\mathbf{x}) \ . \tag{3.1}$$

Furthermore, the metric $\gamma_{ij} = \delta_{(k)(\ell)} e^{(k)}_{\ i} e^{(\ell)}_{\ j}$ is that of a constant curvature space with curvature constant k. In the case of a flat background, we can write

$$e^{(\ell)}_{\ i} = \delta^{(\ell)}_{\ i} \quad \text{for} \ \ k = 0 \ . \tag{3.2}$$

Hereafter, we restrict our consideration to the Einstein-de Sitter background, $k = 0, \Lambda = 0$.

B. The first-order solutions: scalar modes

Linear perturbations are classified into scalar, vector, and tensor modes. In the first-order level, they do not couple with each other, and can be discussed separately. Let us first consider the scalar perturbations. The general form for the linearly perturbed triad in this case is

$$e^{(\ell)}_{\ i} = \delta^{(\ell)}_{\ i} + E^{(\ell)}_{\ i} = \delta^{(\ell)}_{\ i} + \delta^{(\ell)}_{\ j} \left(A \, \delta^j_{\ i} + B^{,j}_{\ ,i} \right) \ . \tag{3.3}$$

Let us write the first-order quantities with subscript (1). Then the perturbed extrinsic curvature is

$$V^i_{(1)\,j} = \dot{A} \, \delta^i_{\ j} + \dot{B}^{,i}_{\ ,j} \ . \tag{3.4}$$

From the constraint equation (2.14), which reads

$$V^i_{(1)\,j,i} - V^i_{(1)\,i,j} = 0 \tag{3.5}$$

in the first-order, we obtain $\dot{A}_{,i} = 0$. However, the part $\dot{A}(t)\,\delta^i{}_j$ in the extrinsic curvature simply represents the uniform and isotropic Hubble expansion. Therefore, by a suitable re-definition of the background, we can set

$$\dot{A} = 0\ , \quad \text{i.e.,} \quad A = A(\mathbf{x})\ . \tag{3.6}$$

As was noted previously, it is apparent that the source terms $P^{(\ell)}_i$ and $Q^{(\ell)}_i$ are second-order quantities (and higher). Using $V^i_{(1)\,j} = \dot{B}{}^{,i}{}_{,j}$, we also find that $S^{(\ell)}_i$ vanishes in linear order:

$$S^{(\ell)}_{(1)\,i} = \delta^{(\ell)}_j \left(V^{j}_{(1)\,i}{}^{,k}{}_{,k} + V^{k}_{(1)\,k}{}^{,j}{}_{,i} - V^{j}_{(1)\,k,i}{}^{,k} - V^{k}_{(1)\,i}{}^{,j}{}_{,k} \right) = 0\ . \tag{3.7}$$

Therefore, to first-order, the right-hand-side of the key equation (2.21) vanishes and it can be integrated to give

$$a^3 \left(\ddot{E}^{(\ell)}_i + 2\frac{\dot{a}}{a}\dot{E}^{(\ell)}_i - 4\pi G\rho_b E^{(\ell)}_i \right) = C^{(\ell)}_i(\mathbf{x})\ . \tag{3.8}$$

By choosing $C^{(\ell)}_i(\mathbf{x}) = -4\pi G\rho_b a^3\,\delta^{(\ell)}_j \left(A(\mathbf{x})\,\delta^j{}_i + C^{,j}{}_{,i}(\mathbf{x}) \right)$, Eq. (3.8) is re-written as

$$\frac{\partial^2}{\partial t^2}\left(B^{,i}{}_{,j} - C^{,i}{}_{,j} \right) + 2\frac{\dot{a}}{a}\frac{\partial}{\partial t}\left(B^{,i}{}_{,j} - C^{,i}{}_{,j} \right) - 4\pi G\rho_b \left(B^{,i}{}_{,j} - C^{,i}{}_{,j} \right) = 0\ . \tag{3.9}$$

Note that now it has the same form as the equation which governs the density contrast δ in conventional linear perturbation theory [20]. Using the growing mode $D^+(t) = a(t) = t^{2/3}$ and the decaying mode solutions $D^-(t) = t^{-1}$ respectively, we obtain the solutions in the form

$$B^{,i}{}_{,j} = C^{,i}{}_{,j}(\mathbf{x}) + t^{\frac{2}{3}}\,\Psi^{,i}{}_{,j}(\mathbf{x}) + t^{-1}\,\Phi^{,i}{}_{,j}(\mathbf{x})\ . \tag{3.10}$$

For the metric, we have the following first-order expression:

$$g_{ij} = a^2(t)\left[(1 + 2A(\mathbf{x}))\,\delta_{ij} + 2C_{,ij}(\mathbf{x}) + 2t^{\frac{2}{3}}\,\Psi_{,ij}(\mathbf{x}) + 2t^{-1}\,\Phi_{,ij}(\mathbf{x}) \right]\ . \tag{3.11}$$

The relation between $A(\mathbf{x})$ and $\Psi(\mathbf{x})$ is given by Eq. (2.15). To first-order, it reads

$$\dot{V}^i_{(1)\,j} + 3\frac{\dot{a}}{a}V^i_{(1)\,j} + \frac{1}{a^2}\left(\mathcal{R}^i_{(1)\,j} - \frac{1}{4}\mathcal{R}^k_{(1)\,k}\delta^i{}_j\right) = 0\ , \tag{3.12}$$

where

$$V^i_{(1)\,j} = \frac{2}{3}t^{-\frac{1}{3}}\Psi^{,i}{}_{,j} - t^{-2}\Phi^{,i}{}_{,j} \tag{3.13}$$

and

$$\mathcal{R}^i_{(1)\,j} = -A^{,i}{}_{,j} - A^{,k}{}_{,k}\,\delta^i{}_j\ . \tag{3.14}$$

Hence we have

$$A(\mathbf{x}) = \frac{10}{9}\Psi(\mathbf{x})\ . \tag{3.15}$$

The function $C(\mathbf{x})$ is not determined by the Einstein equations within our approximation. As shown in APPENDIX A, however, we can set $C(\mathbf{x}) = 0$ using a residual gauge freedom. The final form of the first-order solutions is, therefore,

$$e^{(\ell)}_{\ i} = \left(1 + \frac{10}{9}\Psi(\mathbf{x})\right)\delta^{(\ell)}_{\ i} + \delta^{(\ell)}_{\ j}\left(t^{\frac{2}{3}}\,\Psi^{,j}{}_{,i}(\mathbf{x}) + t^{-1}\,\Phi^{,j}{}_{,i}(\mathbf{x})\right)\ , \tag{3.16}$$

or in the form of the metric

$$g_{ij} = a^2(t)\left[\left(1 + \frac{20}{9}\Psi(\mathbf{x})\right)\delta_{ij} + 2t^{\frac{2}{3}}\,\Psi_{,ij}(\mathbf{x}) + 2t^{-1}\,\Phi_{,ij}(\mathbf{x})\right]\ . \tag{3.17}$$

Note that we have not assumed that the density contrast is small, in order to derive the solutions. The density is given by Eq. (2.7), which in this case reads

$$\rho = \rho(t_{in}, \mathbf{x})\left(\frac{a(t_{in})}{a(t)}\right)^3 \frac{\det[e^{(\ell)}_{\ i}(t_{in}, \mathbf{x})]}{\det[e^{(\ell)}_{\ i}(t, \mathbf{x})]}\ . \tag{3.18}$$

C. The first-order solutions: tensor modes

Under the assumption of vanishing vorticity, the remaining is the tensor mode perturbations. In this case, we can write the triad in the form

$$e^{(\ell)}_{\ i} = \delta^{(\ell)}_{\ i} + \delta^{(\ell)}_{\ j}H^j{}_i \tag{3.19}$$

with $H^i{}_{j,i} = 0$ and $H^i{}_i = 0$.

The perturbed extrinsic curvature is

$$V^i_{(1)j} = \dot{H}^i{}_j \; . \tag{3.20}$$

Then, the constraint equation (2.14) to first-order, i.e., Eq. (3.5), is trivially satisfied.

To obtain the equation for $H^i{}_j$, we can use the key equation (2.21). On the right-hand-side of Eq. (2.21), $S^{(\ell)}_i$ is the only quantity to be calculated since $P^{(\ell)}_i$ and $Q^{(\ell)}_i$ are of higher order:

$$S^{(\ell)}_{(1)\,i} = \delta^{(\ell)}_j \dot{H}^j{}_i{}^{,k}{}_{,k} \; . \tag{3.21}$$

Eq. (2.21) reads

$$\frac{\partial}{\partial t}\left[a^3\left(\ddot{H}^i{}_j + 2\frac{\dot{a}}{a}\dot{H}^i{}_j - 4\pi G\rho_b H^i{}_j\right)\right] = a\,\dot{H}^i{}_j{}^{,k}{}_{,k} \; . \tag{3.22}$$

Integrating Eq. (3.22), we obtain

$$\ddot{H}^i{}_j + 3\frac{\dot{a}}{a}\dot{H}^i{}_j - \frac{1}{a^2}\nabla^2 H^i{}_j = 0 \; , \tag{3.23}$$

where ∇^2 is the Laplacian of flat 3-spaces. In fact, the same equation for $H^i{}_j$ can be also obtained directly from Eq. (3.12). The solution of Eq. (3.23) is given as

$$H^i{}_j = \int d^3\mathbf{q}\;\; t^{-\frac{1}{2}} J_{\pm\frac{3}{2}}(3|\mathbf{q}|t^{\frac{1}{3}})\;\; h^i{}_j \exp(i\mathbf{q}\cdot\mathbf{x}) \; , \tag{3.24}$$

where $J_{\pm\frac{3}{2}}$ is the Bessel function of order $\pm 3/2$ and $h^i{}_j$ is a constant tensor with $h^i{}_i = 0$ and $q_i h^i{}_j = 0$. (See, e.g., Ref. [21] for detail.)

D. The second-order solutions

In order to avoid notational complexity, in this subsection we only deal with growing mode terms. The complete solutions of the decaying and coupling terms can be found in APPENDIX B. Moreover, we omit the first-order tensor mode. (It is not our aim to consider

the nonlinear effect which comes from this mode. With respect to this problem, see Ref. [9,10].) Thus we begin with the following form

$$e^{(\ell)}_{\;i} = \left(1 + \frac{10}{9}\Psi(\mathbf{x})\right)\delta^{(\ell)}_{\;i} + t^{\frac{2}{3}}\,\delta^{(\ell)}_{\;j}\Psi^{,j}_{\;\;,i}(\mathbf{x}) + \varepsilon^{(\ell)}_{\;i}\;. \tag{3.25}$$

The second-order quantity $\varepsilon^{(\ell)}_{\;i}$ is decomposed into a transverse-traceless part and a remaining longitudinal part

$$\varepsilon^{(\ell)}_{\;i} = \delta^{(\ell)}_{\;j}\left(\beta^{j}_{\;i} + \chi^{j}_{\;i}\right)\;, \tag{3.26}$$

where $\chi^{i}_{\;j,i} = 0,\;\chi^{i}_{\;i} = 0.$

The peculiar deformation tensor to second-order is immediately found to give

$$V^{i}_{(2)\,j} = \dot\beta^{i}_{\;j} + \dot\chi^{i}_{\;j} - \frac{20}{27}t^{-\frac{1}{3}}\Psi\Psi^{,i}_{\;\;,j} - \frac{2}{3}t^{\frac{1}{3}}\Psi^{,i}_{\;\;,k}\Psi^{,k}_{\;\;,j}\;. \tag{3.27}$$

Quantities with subscript (2) represent the second-order term in the expansion. From the constraint equation (2.14), we now obtain

$$\dot\beta^{i}_{\;j,i} - \dot\beta^{i}_{\;i,j} + \frac{20}{27}t^{-\frac{1}{3}}\left(\Psi^{,k}\Psi_{,k}\right)_{,j} = 0\;. \tag{3.28}$$

Let us turn our attention to the key equation (2.21). To second-order it reads

$$\ddot\varepsilon^{(\ell)}_{\;i} + 2\frac{\dot a}{a}\dot\varepsilon^{(\ell)}_{\;i} - 4\pi G\rho_b\varepsilon^{(\ell)}_{\;i} = \frac{1}{a^3}c^{(\ell)}_{\;i}(\mathbf{x}) + \frac{1}{a^3}\int^t a\,\left(P^{(\ell)}_{(2)\,i} + Q^{(\ell)}_{(2)\,i} + S^{(\ell)}_{(2)\,i}\right)dt\;, \tag{3.29}$$

where $c^{(\ell)}_{\;i}(\mathbf{x})$ is a second-order integration "constant". It is apparent that the source terms $P^{(\ell)}_{(2)\,i}$ and $Q^{(\ell)}_{(2)\,i}$ are quadratic with respect to the first-order quantities, hence contain neither $\beta^{i}_{\;j}$ nor $\chi^{i}_{\;j}$. Furthermore, from Eq. (3.28), we find that the longitudinal part of $S^{(\ell)}_{(2)\,i}$ does not contain $\beta^{i}_{\;j}$. Actually, if we take the divergence of Eq. (3.29), we obtain

$$\ddot\beta^{i}_{\;j,i} + 2\frac{\dot a}{a}\dot\beta^{i}_{\;j,i} - 4\pi G\rho_b\beta^{i}_{\;j,i} = \frac{1}{a^3}c^{i}_{\;j,i}(\mathbf{x}) - \frac{1}{3}t^{-\frac{2}{3}}\left((\Psi^{,k}_{\;\;,k})^2 - \Psi^{,k}_{\;\;,\ell}\Psi^{,\ell}_{\;\;,k}\right)_{,j}\;. \tag{3.30}$$

Therefore, solutions for $\beta^{i}_{\;j}$ can be written as a linear combination of the homogeneous solution and the inhomogeneous solution in the presence of the given source terms:

$$\beta^{i}_{\;j} = \alpha(\mathbf{x})\,\delta^{i}_{\;j} + t^{\frac{2}{3}}\,\psi^{i}_{\;j}(\mathbf{x}) + t^{\frac{4}{3}}\,\varphi^{i}_{\;j}(\mathbf{x})\;, \tag{3.31}$$

where we have used a convenient choice of the integration "constant", $c^{(\ell)}_i(\mathbf{x}) = -4\pi G\rho_b a^3 \alpha(\mathbf{x})\, \delta^i{}_j$.

Once we obtain the temporal dependency of the solutions, their spatial dependency, i.e., $\psi^i{}_j(\mathbf{x})$ and $\varphi^i{}_j(\mathbf{x})$ are determined by Eq. (2.15). To second-order

$$\psi^i{}_j = \frac{5}{9}\Psi^{,k}\Psi_{,k}\delta^i{}_j - \frac{10}{9}\left(\Psi^2\right)^{,i}{}_{,j} + \frac{9}{10}\alpha^{,i}{}_{,j} \,, \tag{3.32}$$

$$\varphi^i{}_j = \frac{3}{7}\left(\mu^k{}_k\,\delta^i{}_j - 4\mu^i{}_j\right) \,, \tag{3.33}$$

where

$$\mu^i{}_j \equiv \frac{1}{2}\left(\Psi^{,k}{}_{,k}\Psi^{,i}{}_{,j} - \Psi^{,i}{}_{,k}\Psi^{,k}{}_{,j}\right) \,. \tag{3.34}$$

(The tensor $\mu^i{}_j$ has an interesting property: the trace $\mu^i{}_i$ gives the second scalar invariant [1] of the tensor $\Psi^{,i}{}_{,j}$.)

The equation for $\chi^i{}_j$ can also be obtained from Eq. (3.29), but it is more convenient to use Eq. (2.15) instead. To second-order, it gives for the transverse-traceless part

$$\ddot{\chi}^i{}_j + 3\frac{\dot{a}}{a}\dot{\chi}^i{}_j - \frac{1}{a^2}\nabla^2\chi^i{}_j = \mathcal{S}^i{}_j \,, \tag{3.35}$$

where

$$\mathcal{S}^i{}_j = \frac{3}{7}\mu^k{}_k{}^{,i}{}_{,j} + \frac{3}{7}\left(\mu^k{}_k\,\delta^i{}_j - 4\mu^i{}_j\right)^{,\ell}{}_{,\ell} \tag{3.36}$$

is a transverse and traceless tensor: $\mathcal{S}^i{}_i = 0$, $\mathcal{S}^i{}_{j,i} = 0$. This shows that gravitational waves are induced even if there are initially scalar perturbations only. The solution of Eq. (3.35) was given by Tomita [8] in the following way. Introducing the conformal time variable η, which is related to t by $dt = a d\eta$, Eq. (3.35) is rewritten as

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[1] Three scalar invariants of a three-dimensional tensor $A^i{}_j$ are defined by $I(A) \equiv A^i{}_i$, $II(A) \equiv 1/2[(A^i{}_i)^2 - A^i{}_j A^j{}_i]$, $III(A) \equiv \det(A^i{}_j)$. They satisfy the relation $\det(\delta^i{}_j + A^i{}_j) = 1 + I(A) + II(A) + III(A)$. See, e.g., Ref. [6] and references therein.

$$\frac{\partial^2}{\partial\eta^2}\chi^i{}_j + \frac{4}{\eta}\frac{\partial}{d\eta}\chi^i{}_j - \nabla^2\chi^i{}_j = \frac{1}{81}\eta^4\mathcal{S}^i{}_j \ . \tag{3.37}$$

Eq. (3.37) can be solved using the retarded Green function as

$$\chi^i{}_j(\mathbf{x},\eta) = \frac{1}{81}\int D_{\text{ret}}(\mathbf{x},\eta\,;\mathbf{x'},\eta')\ a^4(\eta')\ \mathcal{S}^i{}_j(\mathbf{x'})\ d\eta'\, d^3\mathbf{x'}\ , \tag{3.38}$$

where

$$D_{\text{ret}}(\mathbf{x},\eta\,;\mathbf{x'},\eta') = \frac{1}{4\pi(\eta\eta')^3}\left(1+\epsilon(\eta-\eta')\right)\left(\eta\eta'\delta\left(\tau^2-r^2\right)+\frac{1}{2}\theta\left(\tau^2-r^2\right)\right) \tag{3.39}$$

with $r \equiv |\mathbf{x}-\mathbf{x'}|$ and $\tau \equiv \eta-\eta'$. Substituting Eq. (3.39) into Eq. (3.38), the solution reads

$$\chi^i{}_j(\mathbf{x},\eta) = \frac{1}{1944\pi\eta^3}\int_0^{\eta-\eta_{in}} dr'\ r'\left((6\eta+r')(\eta-r')^6 - r'\eta_{in}^6\right)\int d\Omega'\ \mathcal{S}^i{}_j(\mathbf{x}+\mathbf{x'})\ , \tag{3.40}$$

where $r' \equiv |\mathbf{x'}|$.

Finally, we obtain the metric tensor up to second-order

$$\begin{aligned}\gamma_{ij} = a^{-2}\,g_{ij} &= \left(1+\frac{20}{9}\Psi+\frac{100}{81}\Psi^2+2\alpha\right)\delta_{ij}\\ &+a(t)\left[\left(2\Psi-\frac{20}{9}\Psi^2+\frac{9}{5}\alpha\right)_{,ij}+\frac{20}{9}\Psi\Psi_{,ij}+\frac{10}{9}\Psi^{,k}\Psi_{,k}\,\delta_{ij}\right]\\ &+a^2(t)\left[\frac{19}{7}\Psi^{,k}_{\ ,i}\Psi_{,kj}-\frac{12}{7}\Psi^{,k}_{\ ,k}\Psi_{,ij}+\frac{3}{7}\left(\left(\Psi^{,k}_{\ ,k}\right)^2-\Psi^{,k}_{\ ,\ell}\Psi^{,\ell}_{\ ,k}\right)\delta_{ij}\right]+2\chi_{ij}\ .\end{aligned} \tag{3.41}$$

We still have freedom in choosing $\alpha(\mathbf{x})$, which corresponds to the second-order term of the initial amplitude of the gravitational potential fluctuations. It can be absorbed into the first-order perturbations by a suitable re-definition of $\Psi(\mathbf{x})$. For example, choosing $\alpha = -\frac{50}{81}\Psi^2$ gives

$$\begin{aligned}\gamma_{ij} &= \left(1+\frac{20}{9}\Psi\right)\delta_{ij}+2a(t)\Psi_{,ij}\\ &+\frac{10}{9}a(t)\left(-6\Psi_{,i}\Psi_{,j}-4\Psi\Psi_{,ij}+\Psi^{,k}\Psi_{,k}\,\delta_{ij}\right)\\ &+\frac{1}{7}a^2(t)\left[19\Psi^{,k}_{\ ,i}\Psi_{,kj}-12\Psi^{,k}_{\ ,k}\Psi_{,ij}+3\left(\left(\Psi^{,k}_{\ ,k}\right)^2-\Psi^{,k}_{\ ,\ell}\Psi^{,\ell}_{\ ,k}\right)\delta_{ij}\right]+2\chi_{ij}\ .\end{aligned} \tag{3.42}$$

At the initial time ($t \to 0$) only first-order metric perturbations exist.

IV. COMPARISON WITH PREVIOUS WORKS

In this section, we compare our result (Eq. (3.42)) with previous work. Quantities, which refer to these papers, will be indicated by a hat.

A. Tomita's second-order theory

Tomita [8] extended Lifshitz's linearized theory [7] up to the second-order calculation on the basis of general relativity. Setting $\hat{F} = \frac{20}{9}\Psi$ for the growing mode and $\hat{F} = 54\Phi$ for the decaying mode (see Eq. (4.1) in Ref. [8]) his result is fully coincident with ours, except for one point: he did not consider the terms due to the coupling between the growing and decaying modes, which are included in our complete solutions. (See APPENDIX B.)

B. The fluid flow approach

Matarrese et al. [12] also carried out second-order calculations based on the fluid flow approach. Their result (Eq. (49) in Ref. [12]) is partly consistent with ours, since they neglect several terms in the computed metric. In spite of the fact that they obtain the initial condition from the gauge-invariant linear theory, they neglect the first-order constant mode, $\frac{20}{9}\,\Psi\delta_{ij}$ in our notation in Eq. (3.42) in the subsequent calculations. Also missing is the second-order homogeneous solution, which is proportional to $t^{2/3}$.

The comparison of the second-order transverse-traceless parts has to be taken with some caution. Eq. (B19) in Ref. [12], which has to be solved, can be derived from our Eq. (3.35). In the short-wavelength limit inside the horizon ($\eta \gg r'$) in our approach we get $\nabla^2\chi^i{}_j = -t^{\frac{4}{3}}\mathcal{S}^i{}_j$, which can be identified with Eq. (65) in Ref. [12]. In the long-wavelength limit outside the horizon they obained a result, which can be neglected cause of the appearance of spatial derivatives, whereas in our exact result there exists no solution for the wavelength larger than the horizon size.

C. The gradient expansion technique

Parry et al. [13] derived a nonlinear solution for g_{ij}, based on the gradient expansion method. (See also Ref. [15,16].) Their "fifth-order" result is the

$$\hat{\gamma}_{ij} = t^{\frac{4}{3}}\hat{k}_{ij} + \frac{9}{20}t^2\left(\hat{R}\hat{k}_{ij} - 4\hat{R}_{ij}\right) + \frac{81}{350}t^{\frac{8}{3}}\left[\left(\frac{89}{32}\hat{R}^2 + \frac{5}{8}\hat{R}^{;k}_{\ ;k} - 4\hat{R}^{k\ell}\hat{R}_{k\ell}\right)\hat{k}_{ij} \right.$$
$$\left. - 10\hat{R}\hat{R}_{ij} + \frac{5}{8}\hat{R}_{;ij} + 17\hat{R}_i^{\ n}\hat{R}_{jn} - \frac{5}{2}\hat{R}_{ij;k}^{\ \ \ \ ;k}\right], \tag{4.1}$$

where $\hat{k}_{ij} = \hat{k}_{ij}(\mathbf{x})$ is the "seed" metric, $\hat{R}_{ij}$ and $\hat{R}$ are the 3-dimensional Ricci tensor and Ricci scalar, respectively, of the 3-metric $\hat{k}_{ij}$, and a semicolon (;) denotes the covariant derivative with respect to $\hat{k}_{ij}$. To compare our solution Eq. (3.42) with their Eq. (4.1), we set

$$\hat{k}_{ij}(\mathbf{x}) = \left(1 + \frac{20}{9}\Psi(\mathbf{x})\right)\delta_{ij}\ . \tag{4.2}$$

Then the Ricci tensor up to second-order is

$$\hat{R}_{ij} = -\frac{10}{9}\left(\Psi_{,ij} + \Psi^{,k}_{\ ,k}\,\delta_{ij}\right)$$
$$+ \frac{100}{27}\Psi_{,i}\Psi_{,j} + \frac{200}{81}\Psi\Psi_{,ij} + \left(\frac{100}{81}\Psi^{,k}\Psi_{,k} + \frac{200}{81}\Psi\Psi^{,k}_{\ ,k}\right)\delta_{ij}\ . \tag{4.3}$$

If we substitute this expression into Eq. (4.1) and calculate up to the second-order, we obtain

$$\gamma_{ij} \equiv t^{-\frac{4}{3}}\hat{\gamma}_{ij} = \left(1 + \frac{20}{9}\Psi\right)\delta_{ij} + 2t^{\frac{2}{3}}\Psi_{,ij}$$
$$+ \frac{10}{9}t^{\frac{2}{3}}\left(-6\Psi_{,i}\Psi_{,j} - 4\Psi\Psi_{,ij} + \Psi^{,k}\Psi_{,k}\,\delta_{ij}\right) \tag{4.4}$$
$$+ \frac{1}{7}t^{\frac{4}{3}}\left[19\Psi^{,k}_{\ ,i}\Psi_{,kj} - 12\Psi^{,k}_{\ ,k}\Psi_{,ij} + 3\left(\left(\Psi^{,k}_{\ ,k}\right)^2 - \Psi^{,k}_{\ ,\ell}\Psi^{,\ell}_{\ ,k}\right)\delta_{ij}\right]\ .$$

Therefore, we find that their "fifth-order" result coincides with our second-order solution, except for the transverse-traceless part, χ_{ij}. If we take the long-wavelength limit, χ_{ij} can be neglected, since spatial derivatives are assumed to be quantities of higher order than time derivatives in this limit and as a result, the wave equation for χ_{ij} does not appear. In this sense, our result includes theirs.

V. CONCLUDING REMARKS

In this paper, we have developed the second-order perturbative approach to the nonlinear evolution of irrotational dust universes in the framework of general relativity. We have shown the complete calculation of the second-order solutions in a $k = 0,\ \Lambda = 0$ background, based on the tetrad formalism given by Kasai [17]. As mentioned in Sec. IV, our second-order solution includes the results given in Tomita [8], Matarrese et al. [12] and Parry et al. [13] although the essential calculation we need in our approach is just the solution of a second-order ordinary differential equation by iteration method. Therefore, our approach surpasses these others in perfection and simplicity.

Another advantage of our method remains, which is not mentioned above. Tomita's approach is valid only when the absolute value of the density contrast $|\delta| \ll 1$ while ours does not rely on this assumption, which is the inherent usefulness of the so-called Zel'dovich approximation. The gradient expansion technique implies taking the "square root" of the metric tensor in order to reproduce the Zel'dovich approximation, while we do not need such a trick since we start from the tetrad formalism.

In our approach the extensions to $k \neq 0,\ \Lambda \neq 0$ cases and radiation universes ($p = \frac{1}{3}\rho$) are straightforward. These will be the subjects of future investigation.

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APPENDIX A: GAUGE CONDITION

The most general gauge transformation to first order is the result of the infinitesimal coordinate transformation

$$\tilde{x}^\mu = x^\mu + \xi^\mu \ . \tag{A1}$$

The changes in the 4-velocity and in the metric tensor are computed from

$$\tilde{u}^\mu(\tilde{x}^\lambda) = \frac{\partial \tilde{x}^\mu}{\partial x^\nu} u^\nu(x^\lambda) \ , \qquad g_{\mu\nu}(x^\lambda) = \frac{\partial \tilde{x}^\alpha}{\partial x^\mu}\frac{\partial \tilde{x}^\beta}{\partial x^\nu} \tilde{g}_{\alpha\beta}(\tilde{x}^\lambda) \ , \tag{A2}$$

which gives to first order

$$\delta_G u^\mu \equiv \tilde{u}^\mu(x^\lambda) - u^\mu(x^\lambda) = \xi^\mu_{\ ,\nu} u^\nu - u^\mu_{\ ,\nu}\xi^\nu \ , \tag{A3}$$

$$\delta_G g_{\mu\nu} \equiv \tilde{g}_{\mu\nu}(x^\lambda) - g_{\mu\nu}(x^\lambda) = -g_{\mu\nu,\alpha}\xi^\alpha - g_{\mu\alpha}\xi^\alpha_{\ ,\nu} - g_{\nu\alpha}\xi^\alpha_{\ ,\mu} \ .$$

If the perturbations are linear, we can treat scalar perturbations seperate and we write

$$\xi^\mu = (T, \delta^{ij} L_{,j}) \tag{A4}$$

for the $k = 0$ background, where $T = T(x^\mu)$ and $L = L(x^\mu)$ are arbitrary scalar functions.

The gauge condition we impose in this paper is the comoving synchronous condition:

$$u^i = 0 \ , \quad g_{00} = -1 \ , \quad g_{0i} = 0 \ . \tag{A5}$$

These equations must hold for every gauge transformation, so that $\delta_G u^i = \delta_G g_{00} = \delta_G g_{0i} = 0$ lead to

$$\dot{L}_{,j} = 0 \ , \quad \dot{T} = 0 \ , \quad T_{,i} = 0 \ . \tag{A6}$$

Apart from a trivial constant translation, these are solved to give

$$T = 0 \ , \qquad L_{,j} = L_{,j}(\mathbf{x}) \ . \tag{A7}$$

The change due to the residual gauge freedom is

$$\delta_G g_{ij} = -2a^2 L_{,ij}(\mathbf{x}) \ , \tag{A8}$$

or if we use Eq. (3.11),

$$\delta_G C_{,ij}(\mathbf{x}) = -L_{,ij}(\mathbf{x}) \ . \tag{A9}$$

Therefore, using the remaining gauge freedom $L_{,j}(\mathbf{x})$, we can choose $C_{,ij}(\mathbf{x}) = 0$.

APPENDIX B: COMPLETE SECOND-ORDER SOLUTIONS

The complete solution for the triad reads

$$e^{(\ell)}_{\ i} = \left(1+\frac{10}{9}\Psi\right)\delta^{(\ell)}_{\ i} + \delta^{(\ell)}_{\ j}\left(t^{\frac{2}{3}}\ \Psi^{,j}_{\ \ ,i} + t^{-1}\ \Phi^{,j}_{\ \ ,i}\right)$$
$$+\ \delta^{(\ell)}_{\ j}\left(t^{\frac{2}{3}}\ \psi^{j}_{\ i} + t^{-1}\ \phi^{j}_{\ i} + t^{\frac{4}{3}}\ \varphi^{j}_{\ i} + t^{-\frac{1}{3}}\ \upsilon^{j}_{\ i} + t^{-2}\ \zeta^{j}_{\ i} + \chi^{j}_{\ i} + \vartheta^{j}_{\ i} + \theta^{j}_{\ i}\right)\ , \tag{B1}$$

where $\psi^{i}_{\ j}$ and $\phi^{i}_{\ j}$ are the spatially dependent parts of the second-order homogeneous solutions of Eq. (2.21), $\varphi^{i}_{\ j}$, $\upsilon^{i}_{\ j}$ and $\zeta^{i}_{\ j}$ those of the second-order inhomogeneous solutions, which come from $\Psi\times\Psi$, $\Psi\times\Phi$ and $\Phi\times\Phi$, and $\chi^{i}_{\ j}$, $\vartheta^{i}_{\ j}$ and $\theta^{i}_{\ j}$ are the corresponding transverse-traceless parts.

1. The coupling mode

We obtain

$$\upsilon^{i}_{\ j} = 2\Psi^{,k}_{\ \ ,k}\Phi^{,i}_{\ \ ,j} + 2\Phi^{,k}_{\ \ ,k}\Psi^{,i}_{\ \ ,j} - 19\Psi^{,i}_{\ \ ,k}\Phi^{,k}_{\ \ ,j} - 15\Psi^{,k}\Phi^{,i}_{\ \ ,jk}$$
$$+\left(6\Psi^{,k}_{\ \ ,\ell}\Phi^{,\ell}_{\ \ ,k} - \Psi^{,k}_{\ \ ,k}\Phi^{,\ell}_{\ \ ,\ell} + 5\Psi^{,k}\Phi^{,\ell}_{\ \ ,k\ell}\right)\delta^{i}_{\ j} + \frac{9}{2}\left(\phi^{k\ \ ,i}_{\ k\ \ ,j} - \phi^{i\ \ ,k}_{\ j\ \ ,k}\right) \tag{B2}$$

with

$$\phi^{i}_{\ j,i} - \phi^{i}_{\ i,j} = -\frac{20}{9}\Psi_{,k}\Phi^{,k}_{\ \ ,j}\ . \tag{B3}$$

(In this calculation we use $\Psi^{,i}_{\ \ ,k}\Phi^{,k}_{\ \ ,j} = \Phi^{,i}_{\ \ ,k}\Psi^{,k}_{\ \ ,j}$, which comes from $V^{i}_{j} \equiv \frac{1}{2}\gamma^{ik}\dot{\gamma}_{jk} = e^{i}_{(\ell)}\dot{e}^{(\ell)}_{\ j}$.)

The equation for the transverse-traceless part is

$$\ddot{\vartheta}^{i}_{\ j} + 3\frac{\dot{a}}{a}\dot{\vartheta}^{i}_{\ j} - \frac{1}{a^2}\nabla^2\vartheta^{i}_{\ j} = t^{-\frac{5}{3}}\mathcal{P}^{i}_{\ j}\ , \tag{B4}$$

where

$$\mathcal{P}^{i}_{\ j} = \left(6\Psi^{,k}_{\ \ ,\ell}\Phi^{,\ell}_{\ \ ,k} - \Psi^{,k}_{\ \ ,k}\Phi^{,\ell}_{\ \ ,\ell} + 5\Psi^{,k}\Phi^{,\ell}_{\ \ ,k\ell}\right)^{,i}_{\ \ ,j}$$
$$+\Big[\,2\Psi^{,k}_{\ \ ,k}\Phi^{,i}_{\ \ ,j} + 2\Phi^{,k}_{\ \ ,k}\Psi^{,i}_{\ \ ,j} - 24\Psi^{,i}_{\ \ ,k}\Phi^{,k}_{\ \ ,j} - 20\Psi^{,k}\Phi^{,i}_{\ \ ,jk}$$
$$+\left(6\Psi^{,k}_{\ \ ,\ell}\Phi^{,\ell}_{\ \ ,k} - \Psi^{,k}_{\ \ ,k}\Phi^{,\ell}_{\ \ ,\ell} + 5\Psi^{,k}\Phi^{,\ell}_{\ \ ,k\ell}\right)\delta^{i}_{\ j} + \frac{9}{2}\left(\phi^{k\ \ ,i}_{\ k\ \ ,j} - \phi^{i\ \ ,k}_{\ j\ \ ,k}\right)\Big]^{,m}_{\ \ ,m}\ . \tag{B5}$$

Using the conformal time η, this is rewritten as

$$\frac{\partial^2}{\partial\eta^2}\vartheta^i{}_j + \frac{4}{\eta}\frac{\partial}{\partial\eta}\vartheta^i{}_j - \nabla^2\vartheta^i{}_j = \frac{3}{\eta}\mathcal{P}^i{}_j \ . \tag{B6}$$

The solution is

$$\vartheta^i{}_j(\mathbf{x},\eta) = \frac{3}{4\pi\eta^3}\int_0^{\eta-\eta_{in}} dr'\ r'\left((\eta+r')(\eta-r') - r'\eta_{in}\right)\int d\Omega'\ \mathcal{P}^i{}_j(\mathbf{x}+\mathbf{x'}) \ . \tag{B7}$$

2. The decaying mode

We obtain

$$\zeta^i{}_j = \frac{1}{4}\left(\lambda^k{}_k\,\delta^i{}_j - 4\lambda^i{}_j\right) \ , \tag{B8}$$

where

$$\lambda^i{}_j \equiv \frac{1}{2}\left(\Phi^{,k}{}_{,k}\Phi^{,i}{}_{,j} - \Phi^{,i}{}_{,k}\Phi^{,k}{}_{,j}\right) \ . \tag{B9}$$

The equation for the transverse-traceless part is

$$\ddot{\theta}^i{}_j + 3\frac{\dot{a}}{a}\dot{\theta}^i{}_j - \frac{1}{a^2}\nabla^2\theta^i{}_j = t^{-\frac{10}{3}}\mathcal{Q}^i{}_j \ , \tag{B10}$$

where

$$\mathcal{Q}^i{}_j = \frac{1}{4}\lambda^k{}_k{}^{,i}{}_{,j} + \frac{1}{4}\left(\lambda^k{}_k\,\delta^i{}_j - 4\lambda^i{}_j\right)^{,\ell}{}_{,\ell} \ . \tag{B11}$$

Again this is rewritten as

$$\frac{\partial^2}{\partial\eta^2}\theta^i{}_j + \frac{4}{\eta}\frac{\partial}{\partial\eta}\theta^i{}_j - \nabla^2\theta^i{}_j = \frac{729}{\eta^6}\mathcal{Q}^i{}_j \tag{B12}$$

and we obtain the solution

$$\theta^i{}_j(\mathbf{x},\eta) = \frac{729}{16\pi\eta^3}\int_0^{\eta-\eta_{in}} dr'\ r'\left((4\eta-r')(\eta-r')^{-4} + r'\eta_{in}^{-4}\right)\int d\Omega'\ \mathcal{Q}^i{}_j(\mathbf{x}+\mathbf{x'}) \ . \tag{B13}$$

3. The complete expression of the metric tensor

The complete metric reads

$$
\begin{aligned}
\gamma_{ij} = {} & \left(1+\frac{20}{9}\Psi\right)\delta_{ij} + 2t^{\frac{2}{3}}\Psi_{,ij} + 2t^{-1}\Phi_{,ij} \\
& + \frac{10}{9}t^{\frac{2}{3}}\left(-6\Psi_{,i}\Psi_{,j} - 4\Psi\Psi_{,ij} + \Psi^{,k}\Psi_{,k}\,\delta_{ij}\right) \\
& + \frac{1}{7}t^{\frac{4}{3}}\left[19\Psi^{,k}_{\ \ ,i}\Psi_{,kj} - 12\Psi^{,k}_{\ \ ,k}\Psi_{,ij} + 3\left(\left(\Psi^{,k}_{\ \ ,k}\right)^2 - \Psi^{,k}_{\ \ ,\ell}\Psi^{,\ell}_{\ \ ,k}\right)\delta_{ij}\right] + 2\chi_{ij} \\
& + 2t^{-1}\left(\phi_{ij} + \frac{10}{9}\Psi\Phi_{,ij}\right) \\
& + t^{-2}\left[2\Phi^{,k}_{\ \ ,i}\Phi_{,kj} - \Phi^{,k}_{\ \ ,k}\Phi_{,ij} + \frac{1}{4}\left((\Phi^{,k}_{\ \ ,k})^2 - \Phi^{,k}_{\ \ ,\ell}\Phi^{,\ell}_{\ \ ,k}\right)\delta_{ij}\right] + 2\theta_{ij} \\
& + t^{-\frac{1}{3}}\Big[4\Psi^{,k}_{\ \ ,k}\Phi_{,ij} + 4\Phi^{,k}_{\ \ ,k}\Psi_{,ij} - 18\Psi^{,k}_{\ \ ,i}\Phi_{,kj} - 18\Phi^{,k}_{\ \ ,i}\Psi_{,kj} - 30\Psi_{,k}\Phi^{,k}_{\ \ ,ij} \\
& \qquad + \left(12\Psi^{,k}_{\ \ ,\ell}\Phi^{,\ell}_{\ \ ,k} - 2\Psi^{,k}_{\ \ ,k}\Phi^{,\ell}_{\ \ ,\ell} + 10\Psi^{,k}\Phi^{,\ell}_{\ \ ,k\ell}\right)\delta_{ij} + 9\left(\phi^{k}_{\ k,ij} - \phi_{ij\ \ ,k}^{\ \ ,k}\right)\Big] + 2\vartheta_{ij}\ .
\end{aligned}
\tag{B14}
$$

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